

The Functional of Super Riemann Surfaces – A “Semi-Classical” Survey

Enno Keßler¹ · Jürgen Tolksdorf¹

Received: 15 July 2015 / Accepted: 8 January 2016 / Published online: 18 February 2016
© The Author(s) 2016. This article is published with open access at Springerlink.com

Abstract This article provides a brief discussion of the functional of super Riemann surfaces from the point of view of classical (i.e., not “super-”) differential geometry. The discussion is based on symmetry considerations and aims to clarify the “borderline” between classical and super differential geometry with respect to the distinguished functional that generalizes the action of harmonic maps and is expected to play a basic role in the discussion of “super Teichmüller space”. The discussion is also motivated by the fact that a geometrical understanding of the functional of super Riemann surfaces from the point of view of super geometry seems to provide serious issues to treat the functional analytically.

Keywords Clifford modules · Dirac operators · Torsion · Non-linear σ -models · Super Riemann surfaces · Supersymmetry

Mathematics Subject Classification (2010) 58A50 · 30F15 · 32G15

1 Introduction

In this section, we give a brief account on the functional of “super harmonic maps”, which will be discussed afterwards in some detail from a classical differential geometrical setting in view of Dirac operators and symmetry considerations. Accordingly, in this section, we are intentionally sketchy with the aim to only provide some motivation for what is following

In honor of Prof. E. Zeidler’s 75th birthday.

✉ Jürgen Tolksdorf
Juergen.Tolksdorf@mis.mpg.de

Enno Keßler
kessler@mis.mpg.de

¹ Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany

without cluttering the brief outline with too many technical details. A more rigorous and clear exposition, especially with respect to the mappings and geometrical constructions, as well as the notation used in this section, will be postponed to subsequent sections.

To begin with, let (M_k, g_k) be Riemannian manifolds ($k = 1, 2$) where M_1 is assumed to be closed compact and orientable. Also, let $\varphi: M_1 \rightarrow M_2$ be a smooth mapping.

The *functional of harmonic maps*

$$\mathcal{I}_H(\varphi) := \int_{M_1} \|d\varphi\|^2 \, d\text{vol}(g_1) \quad (1)$$

is known to play a basic role in geometric analysis. Here,

$$\|\alpha \otimes w\|^2(t) := g_1^*|_t(\alpha, \alpha) g_2|_{\varphi(t)}(w, w)$$

for all $\alpha \otimes w \in \Omega^1(M_1, \varphi^*TM_2)$, where g_1^* denotes the metric on the co-tangent bundle of M_1 . The study of the functional (1) has a long history, actually (e.g., see [6, 7] and the references therein).

In fact, for $M_1 := [0, 1] \subset \mathbb{R}$, the minimizers of (1) are the geodesics of (M_2, g_2) . Furthermore, the functional of harmonic maps plays a basic role in the analytical treatment of minimal surfaces and generalizations thereof.

A particularity of the case where M_1 is two-dimensional is that the functional (1) is not only diffeomorphism invariant but also conformally invariant. This crucial feature allows to apply the action functional (1) to the study of Teichmüller space, which is a contractible covering of the moduli space of compact one-dimensional complex manifolds. For instance, for harmonic maps between surfaces, the “energy-momentum tensor”, defined by the variation of (1) with respect to g_1 , can be geometrically interpreted as a tangent vector to the Teichmüller space at the point M_1 . Moreover, it can be shown using (1) that the tangent space to the Teichmüller space at M_1 can be identified with holomorphic quadratic differentials on M_1 .

The functional (1) has then been extended, for instance, by including spinors, to the *functional of Dirac harmonic maps* (cf. [2, 3])

$$\mathcal{I}_{\text{DH}}(\varphi, \psi) := \int_{M_1} (\|d\varphi\|^2 + \langle \psi, \not{D}\psi \rangle_{\mathcal{E}}) \, d\text{vol}(g_1). \quad (2)$$

Here, the field ψ is a section of the twisted spinor bundle $\pi_{\mathcal{E}}: \mathcal{E} := \mathbb{S} \otimes \varphi^*TM_2 \rightarrow M_1$. Consequently, one has to assume that M_1 is a spin manifold, and there is a fiber metric $\langle \cdot, \cdot \rangle_{\mathbb{S}}$ on a corresponding spinor bundle $\pi_{\mathbb{S}}: \mathbb{S} \rightarrow M_1$, such that the Dirac operator is symmetric. The Dirac operator \not{D} is the “quantized” Clifford connection $\nabla^{\mathcal{E}}$ on the twisted spinor bundle \mathcal{E} that arises from the Levi-Civita connections on (M_1, g_1) and (M_2, g_2) , respectively. Also, the fiber metric reads $\langle \cdot, \cdot \rangle_{\mathcal{E}} \equiv \langle \cdot, \cdot \rangle_{\mathbb{S}} \otimes \varphi^*g_2$.

In the study of Dirac harmonic maps, it is decisive to use representations of the Clifford algebra $Cl_{0,2}$ to construct the Dirac operator. Otherwise, the real-valued “Dirac action”

$$\langle\langle \psi, \not{D}\psi \rangle\rangle := \int_{M_1} \langle \psi, \not{D}\psi \rangle_{\mathcal{E}} \, d\text{vol}(g_1) \quad (3)$$

vanishes in the case where $\dim M_1 = 2$. Contrary to what is custom in the study of Dirac harmonic maps, we will work with representations of $Cl_{2,0}$ and look at other ways to prevent the vanishing of the Dirac action (3). The major challenge is that the Clifford algebra $Cl_{2,0}$ of the Euclidean plane $\mathbb{R}^{2,0}$ has no skew-symmetric representation on the underlying real spinor module. In physics, real spinors are called *Majorana spinors*. We adopt this terminology. The consequences of the vanishing of (3) in case of Majorana spinors will be explained in some detail in what follows. Especially, we shall discuss the usual way out of this flaw by the assumption of *odd* (“anti-commuting”) spinors (see below). Indeed, the functional (3)

becomes non-trivial even for real anti-commuting Majorana spinors on Riemann surfaces. Furthermore, incorporating anti-commuting spinors, the action functional (2) is not only invariant under diffeomorphisms and conformal re-scaling of the metric g_1 (i.e., “Weyl transformations”) but also with respect to the variations of (2) that are defined to first order by so-called *supersymmetry transformations*

$$\delta_{\mathfrak{s}}\varphi = \langle \mathfrak{s}, \psi \rangle_{\mathbb{S}} \in \Gamma(M_1, \varphi^*TM_2), \quad (4)$$

$$\delta_{\mathfrak{s}}\psi = \gamma(d\varphi)\mathfrak{s} \in \Gamma(M_1, \mathcal{E}). \quad (5)$$

Here, the variational spinor fields $\mathfrak{s} \in \Gamma_{\text{hol}}(M, \mathbb{S})$ are restricted to arbitrary (odd) *holomorphic* sections.

To avoid this restriction will bring us eventually to the even more enhanced (real-valued) *functional of super harmonic maps*

$$\begin{aligned} \mathcal{I}_{\text{SDH}}(\varphi, \psi, \chi, g_1) := & \int_{M_1} (\|d\varphi\|^2 + \langle \psi, \not{D}\psi \rangle_{\mathcal{E}} + \langle \chi, \mathfrak{q}(\chi) \rangle_{T^*M_1 \otimes \mathbb{S}} \langle \psi, \psi \rangle_{\mathcal{E}} \\ & + 4\langle \mathfrak{q}(\chi)(\text{grad } \varphi), \psi \rangle_{\mathcal{E}}) \text{dvol}(g_1), \end{aligned} \quad (6)$$

with the section $\chi \in \Omega^1(M_1, \mathbb{S})$ being termed *gravitino*. Again, the notation used will be explained in more detail in the next section. Notice also that the functional of super harmonic maps is older than the functional of Dirac harmonic maps as it has been studied already in the seventies in the context of nonlinear super symmetric sigma models and string theory (see [1, 4]).

Besides, the diffeomorphism and conformal invariance (for $\dim M_1 = 2$), the functional (6) carries an additional symmetry. In fact, it is also invariant with respect to *super Weyl transformations*

$$\chi(v) \mapsto \chi(v) + \gamma(v^{\flat})\mathfrak{s}, \quad (7)$$

for all tangent vectors $v \in TM_1$ and arbitrary spinor field $\mathfrak{s} \in \Gamma(M_1, \mathbb{S})$.

In the *two-dimensional case* and for *odd spinors*, the functional (6) has the crucial property that it does not depend on the metric connection on M_1 appearing in the Clifford connection $\nabla^{\mathcal{E}}$. One may therefore replace \not{D} by the Dirac operator \not{D}_A with *torsion*. As a consequence, the functional (6) becomes invariant also with respect to the *enhanced supersymmetry transformations*

$$\delta_{\mathfrak{s}}\varphi := \langle \mathfrak{s}, \psi \rangle_{\mathbb{S}} \in \Gamma(M_1, \varphi^*TM_2), \quad (8)$$

$$\delta_{\mathfrak{s}}\psi := \gamma(d\varphi - \langle \psi, \chi \rangle_{\mathbb{S}})\mathfrak{s} \in \Gamma(M_1, \mathcal{E}), \quad (9)$$

$$\delta_{\mathfrak{s}}e_k := -4\langle \delta_{\Theta}\mathfrak{s}, \chi(e_k) \rangle_{\mathbb{S}} \in \Gamma(M_1, TM_1) \quad (k = 1, 2), \quad (10)$$

$$\delta_{\mathfrak{s}}\chi := d_A\mathfrak{s} \in \Omega^1(M_1, \mathbb{S}), \quad (11)$$

provided g_2 is flat and the torsion factorizes by the gravitino as

$$A = \langle \gamma(\chi), \chi \rangle_{\mathbb{S}} \in \Omega^1(M_1). \quad (12)$$

Here, $e_1, e_2 \in \Gamma(M_1, TM_1)$ denotes an arbitrary (oriented) local g_1 -orthonormal frame.

Notice that in contrast to (4, 5), the supersymmetry transformations (8)–(11) are generated by a completely arbitrary (odd) variational spinor field $\mathfrak{s} \in \Gamma(M_1, \mathbb{S})$. The somewhat simplifying assumption of g_2 being flat can be omitted, actually, by adding an appropriate

curvature term to the integrand of (6). In what follows, however, we restrict ourselves to the case where (M_2, g_2) is flat to keep things more straightforward.

What is the geometrical meaning of the somewhat ad hoc looking functional (6) and how is it related to the functional (1) of Dirac harmonic maps? For more than 30 years, there is the conjecture that the functional of super harmonic mappings (6) is related to the moduli space of *super Riemann surfaces* similar to how the functional of harmonic maps (1) is related to the moduli space of Riemann surfaces (see however [5]).

Recall that a super Riemann surface is a complex super manifold \mathcal{M} of dimension $1|1$ together with a rank $0|1$ dimensional distribution $\mathcal{D} \subset T\mathcal{M}$, such that $\mathcal{D} \otimes \mathcal{D} \simeq_{\mathbb{C}} T\mathcal{M}/\mathcal{D}$. Furthermore, in [9], it has been shown how the functional (6) of super harmonic maps can be re-written on super Riemann surfaces as

$$\mathcal{I}_{\text{SDH}}(\Phi) = \int_{\mathcal{M}} \|d\Phi|_{\mathcal{D}}\|^2 [d\text{vol}], \quad (13)$$

where $\Phi: \mathcal{M} \rightarrow \mathcal{N}$ is a mapping between families of super manifolds which is the analogue of $\varphi: M_1 \rightarrow M_2$ in the case of (1). The formal similarity between (13) and (1) is apparent. Within the super setting, the supersymmetry transformations (8)–(11) have the geometrical meaning of a specific (infinitesimal) diffeomorphism on the super Riemann surface \mathcal{M} . For details, we refer to [8] and, especially, to [9]. There, it is shown, for the first time, in all details that the functional (13) is indeed well-defined on the moduli space of super Riemann surfaces like (1) is well-defined on Teichmüller space. Moreover, the geometrical meaning of the gravitino is shown to be related to the embedding of an underlying Riemann surface into \mathcal{M} . Finally, the variation of (6) with respect to the gravitino is demonstrated to have the geometrical meaning of a tangent vector field on the moduli space of super Riemann surfaces in “odd” directions, similar to the energy-momentum tensor is known to be tangent to the “even” directions.

In a way the geometrical meaning of the functional (6) has been fully clarified in terms of super differential geometry. Yet, one may pose the question to what extent the functional (6) and super Riemann surfaces can also be motivated within the setting of “classical” (i.e., non-super) differential geometry. In fact, in [8], it is discussed how super Riemann surfaces are related to spinor bundles over Riemann surfaces together with the gravitino field χ . This classical geometrical background seems more suitable for geometrical analysis than the super setting. One reason for this might be the fact that within the super setting the functional (6) is not longer real-valued but has to be understood as a mapping between super manifolds. Actually, this seems to be unavoidable when one insists on supersymmetry also within the classical frame as it is exposed below.

In this work, we review on how much of the classical setup can be retained to understand the functional of super harmonic maps (6) together with the super Weyl and supersymmetry transformations (7)–(11). Although much is known on this matter, already, it still seems worth presenting a detailed account on how supersymmetry enforces super geometry. In fact, our discussion should be understood as being complementary to what has been presented in [8] and, in particular, in [9].

2 The Geometrical Setup

In this section, we summarize the geometrical background and fix the notation already used in the previous section. The assumption that M_1 is two-dimensional and orientable has far reaching consequences.

Let (M_1, g_1) be a two-dimensional orientable Riemannian manifold (Riemann surface). The induced Riemannian volume form $\omega_1 \equiv d\text{vol}(g_1) \in \Omega^2(M_1)$ yields an almost complex structure $\mathfrak{J} \in \text{End}(TM_1)$ via

$$g_1(\mathfrak{J}u, v) := \omega_1(u, v) \quad (u, v \in TM_1). \quad (14)$$

Of course, the almost complex structure would be the same for the metric rescaled by a positive function. In other words, there is a one-to-one correspondence between almost complex structures and conformal classes of metrics on M_1 . This well-known fact can also be inferred from the fact that both almost complex structures and conformal classes of metrics on M_1 yield the same reduction of the structure group of the frame bundle by $\mathbb{R}_+ \times \text{SO}(2) \subset \text{GL}(2, \mathbb{R})$.

It is particular to the two-dimensional case that any such almost complex structure is integrable. Consequently, M_1 may be regarded as a one-dimensional complex manifold. It follows that the complexified tangent bundle splits into the holomorphic and anti-holomorphic vector fields on M_1 . That is,

$$TM_1^{\mathbb{C}} \equiv TM_1 \otimes \mathbb{C} \simeq T^{(1,0)}M_1 \oplus T^{(0,1)}M_1. \quad (15)$$

Clearly, the realification of $T^{(1,0)}M_1$ is but TM_1 .

As $\text{Spin}(2) \simeq \text{SO}(2)$, the real rank two *Majorana spinor bundle* $\pi_{\mathbb{S}}: \mathbb{S} \rightarrow M_1$ associated to a chosen spin structure on M_1 is also equipped with a symmetric scalar product and an antisymmetric bilinear form. Those are given by the lifts of the metric g_1 and the volume form ω_1 on TM_1 and will be denoted by the same symbol on \mathbb{S} . Consequently, similar to (14), there is an almost complex structure \mathfrak{J} on \mathbb{S} that depends only on the conformal class of g_1 .

The complexification $S = \mathbb{S} \otimes \mathbb{C}$ is called the complex *bundle of Dirac spinors* and splits into the eigenspaces of \mathfrak{J} of eigenvalue $\pm i$, such that

$$S \equiv \mathbb{S} \otimes \mathbb{C} = W \oplus \overline{W}. \quad (16)$$

The mutually complex conjugate sub-vector bundles W (eigenvalue $+i$) and \overline{W} (eigenvalue $-i$) are called *Weyl spinor bundles*. The complex line bundle W is isomorphic as a complex line bundle to \mathbb{S} , where the complex structure on the latter is given by the almost complex structure \mathfrak{J} . Furthermore, as the chosen spin structure consists of a fiber wise double cover $\text{Spin}(2) \rightarrow \text{SO}(2)$, one obtains the following identities of complex line bundles:

$$W \otimes W \simeq T^{(1,0)}M_1, \quad (17)$$

$$\overline{W} \otimes \overline{W} \simeq T^{(0,1)}M_1. \quad (18)$$

In fact, the choice of a line bundle W with the property (17) is equivalent to the choice of a spin structure. A corollary of (17) is that W is a holomorphic line bundle on the complex manifold M_1 .

Clearly, the factorization of the holomorphic vector fields on a Riemann surface into spinors (17) corresponds to the basic property of a super Riemann surface. That is, the choice of a $0|1$ -distribution $\mathcal{D} \subset \mathcal{TM}$ in the “super world” reduces on the “classical” side to the choice of a spin structure on M_1 . Indeed, one can show that trivial families of super Riemann surfaces are in one-to-one correspondence to Riemann surfaces with chosen spin structure, see, e.g., [10, Proposition 4.2.2.]. To also capture non-trivial families of super Riemann surfaces, one needs the gravitino field χ that appeared already in the first section.

We will now turn to the study of differential forms with spinor values, the bundle of which χ is a section.

The twisted Dirac spinor bundle $\pi_{S \otimes T^* M_1} : S \otimes T^* M_1^{\mathbb{C}} \rightarrow M_1$ decomposes as

$$\begin{aligned} S \otimes T^* M_1^{\mathbb{C}} &\simeq S \oplus (W^3 \oplus \overline{W}^3) \\ &= (\mathbb{S} \oplus \mathbb{G}) \otimes \mathbb{C}. \end{aligned} \quad (19)$$

Here, the real sub-bundle $\pi_{S \otimes \mathbb{G}} : \mathbb{S} \otimes \mathbb{G} \rightarrow M_1$ refers to the canonical real structure on $S \otimes T^* M_1^{\mathbb{C}}$. Notice that the rank two complex vector bundle with total space $W^3 \oplus \overline{W}^3$ has a canonical real structure \mathbb{G} in contrast to the complex line bundle with total space W (or \overline{W}). The latter has a real structure if and only if it is trivial. Notice that \mathbb{G} is the realification of the complex line bundle $W^3 \equiv W \otimes W \otimes W$.

As an upshot also the real twisted spinor bundle $\pi_{S \otimes T^* M_1} : \mathbb{S} \otimes T^* M_1 \rightarrow M_1$ becomes \mathbb{Z}_2 -graded

$$\mathbb{S} \otimes T^* M_1 \simeq \mathbb{S} \oplus \mathbb{G}. \quad (20)$$

Explicitly, the corresponding projection operators read

$$\begin{aligned} \mathfrak{p} : \mathbb{S} \otimes T^* M_1 &\longrightarrow \mathbb{S} \otimes T^* M_1 \\ \sigma_k \otimes e^k &\longmapsto \frac{1}{2} g_1(e_i, e_j) \gamma(e^j) \gamma(e^k) \sigma_k \otimes e^i, \\ \mathfrak{q} : \mathbb{S} \otimes T^* M_1 &\longrightarrow \mathbb{S} \otimes T^* M_1 \\ \sigma_k \otimes e^k &\longmapsto \frac{1}{2} g_1(e_i, e_j) \gamma(e^k) \gamma(e^j) \sigma_k \otimes e^i. \end{aligned}$$

Here, and in the sequel, we take advantage of Einstein's summation convention. Also, e_1, e_2 is a local (oriented) frame on $T M_1$ with dual frame denoted by e^1, e^2 . Finally,

$$\begin{aligned} \gamma : T^* M_1 &\longrightarrow \text{End}(\mathbb{S}) \\ \alpha &\longmapsto \gamma(\alpha) \end{aligned} \quad (21)$$

denotes a *Clifford map*. We also make use of the common notation: $\gamma^k \equiv \gamma(e^k)$, whenever e^1, e^2 is an oriented orthonormal basis with respect to g_1^* .

In more abstract terms, the complementary projection operators \mathfrak{p} and $\mathfrak{q} = 1 - \mathfrak{p}$ are given by the following two mappings:

$$\begin{aligned} \delta_\gamma : \mathbb{S} \otimes T^* M_1 &\longrightarrow \mathbb{S} \\ \sigma_k \otimes e^k &\longmapsto \gamma^k \sigma_k, \\ \delta_\Theta : \mathbb{S} &\longrightarrow \mathbb{S} \otimes T^* M_1 \\ \sigma &\longmapsto \frac{1}{2} \delta_{ij} \gamma^i \sigma \otimes e^j. \end{aligned}$$

Since $\delta_\gamma \circ \delta_\Theta = 1$, one may define $\mathfrak{p} := \delta_\Theta \circ \delta_\gamma$. We call δ_γ the *quantization map* and simply write $\delta_\gamma(\alpha) \equiv \gamma(\alpha) \in \text{End}(\mathbb{S})$ for all $\alpha \in T^* M_1$.

Notice that the projection operators \mathfrak{p} and \mathfrak{q} are self-adjoint, such that the decomposition

$$\mathbb{S} \otimes T^* M_1 = \mathfrak{p}(\mathbb{S} \otimes T^* M_1) \oplus \mathfrak{q}(\mathbb{S} \otimes T^* M_1) \quad (22)$$

becomes orthogonal.

Let $\sharp/\flat : T^* M_1 \simeq T M_1$ be the “musical” isomorphisms defined by g_1 and its dual g_1^* , such that, for instance, $g_1^*(\alpha, \beta) = g_1(\alpha^\sharp, \beta^\sharp)$ for all $\alpha, \beta \in T^* M_1$. We define for all $\mathfrak{s} \in \mathbb{S}$ the canonical inclusion $\mathbb{S} \hookrightarrow \mathbb{S} \otimes T M_1$ by $\delta_{\Theta^\sharp} \mathfrak{s} := \frac{1}{2} \gamma^k \mathfrak{s} \otimes e_k \in \mathbb{S} \otimes T M_1$. Notice that

every Clifford map (21) induces a canonical one-form $\Theta \in \Omega^1(M_1, \text{End}(\mathbb{S}))$ that is given by $\Theta(v) := \frac{1}{2}\gamma(v^b)$ for all $v \in TM_1$. Explicitly, $\Theta = \frac{1}{2}\delta_{ij}\gamma^i \otimes e^j$. Accordingly, we put $\Theta^\sharp := \frac{1}{2}\gamma^k \otimes e_k$.

Any g_1 -orthonormal frame e_1, e_2 for TM_1 yields the hermitian frames $e = (e_1 - ie_2)/\sqrt{2}$ of $T^{(1,0)}M_1$ and $\bar{e} = (e_1 + ie_2)/\sqrt{2}$ for $T^{(0,1)}M_1$. Similarly on \mathbb{S} , a g_1 -orthonormal frame s_1, s_2 gives rise to a hermitian frame $\mathfrak{w} = (s_1 - is_2)/\sqrt{2}$ on W and $\bar{\mathfrak{w}} = (s_1 + is_2)/\sqrt{2}$ on \bar{W} . We suppose furthermore that the frame for S covers the frame for TM_1 , i.e., $\mathfrak{w} \otimes \mathfrak{w} = e$. For the dual spaces $(T^{(1,0)}M_1)^*$ and $(T^{(0,1)}M_1)^*$, we use the dual basis of e, \bar{e} , denoted by θ and $\bar{\theta}$, respectively. Then, by construction of (19), it holds that

$$\begin{aligned} \mathfrak{p}(\mathbb{S} \otimes T^*M_1) &= \{z\mathfrak{w} \otimes \theta + \bar{z}\bar{\mathfrak{w}} \otimes \bar{\theta} \mid z \in \mathbb{C}\} \simeq \mathbb{S}, \\ \mathfrak{q}(\mathbb{S} \otimes T^*M_1) &= \{z\mathfrak{w} \otimes \bar{\theta} + \bar{z}\bar{\mathfrak{w}} \otimes \theta \mid z \in \mathbb{C}\} \simeq \mathbb{G}. \end{aligned} \quad (23)$$

The orthogonal decomposition (20) is but the irreducible decomposition of the twisted Majorana bundle $\mathbb{S} \otimes T^*M_1$ into its spin-1/2 and a spin-3/2 parts. That is, every $\mathfrak{z} \in \mathbb{S} \otimes T^*M_1$ has a unique decomposition

$$\mathfrak{z} = \delta_\Theta \mathfrak{s} + \mathfrak{g},$$

where $\mathfrak{s} \in \mathbb{S}$ is uniquely determined by $\mathfrak{s} := \delta_\gamma(\mathfrak{z})$. Likewise, the spin-3/2 spinor $\mathfrak{g} \in \mathbb{G}$ is uniquely determined by $\delta_\gamma(\mathfrak{g}) := 0$.

It is amazing that the classical realm discussed so far can be basically subsumed by the fact that the Clifford algebra $Cl_{2,0}$ of the Euclidean plane $\mathbb{R}^{2,0}$ decomposes as

$$Cl_{2,0} \simeq \mathbb{C} \oplus \mathbb{R}^{2,0} \quad (24)$$

and by the equality $\text{Spin}(2) = \text{SO}(2)$. The latter identification allows to regard both the metric g_1 and the symplectic form $\omega_1 \equiv d\text{vol}(g_1)$ as being inner products on \mathbb{S} . That is, the notation $\langle \cdot, \cdot \rangle_{\mathbb{S}}$ for the metric on the spinor bundle is but g_1 . The different notation used is just to indicate on whether g_1 acts as an inner product on spinors or on tangent vectors.

As mentioned already, in the usual approach to the action of Dirac harmonic maps (2), one considers the Clifford algebra $Cl_{0,2}$, instead of $Cl_{2,0}$. This is to avoid the flaw of a vanishing Dirac action. Here, one takes into account that $Cl_{0,2} \simeq_{\mathbb{R}} \mathbb{H}$ and identifies the latter, as a vector space, with \mathbb{C}^2 . Notice that the only spinor module of $Cl_{0,2}$ is given by the Clifford algebra $Cl_{0,2}$ itself. Hence, in the usual approach to Dirac harmonic maps on Riemann surfaces, one identifies spinors with sections of a complex vector bundle of rank two. Clearly, this spinor module carries a skew-hermitian representation of the Clifford action. However, in this approach, one loses much of the structure contained in the decomposition (24). In particular, one loses the factorization (17), which is at the very heart of the definition of super Riemann surfaces and the notion of gravitinos. The meaning of the latter within the classical realm will be discussed next.

3 Torsion on Riemann Surfaces

With a connection ∇ on the tangent bundle of an arbitrary smooth manifold M , there are associated two different geometrical objects: the curvature and the *torsion* of this connection. The torsion may be defined as

$$\tau_\nabla := d_\nabla \mathfrak{I}\mathfrak{D},$$

where, respectively, d_∇ and $\mathfrak{I}\mathfrak{D} \in \Omega^1(M, TM)$ are the exterior covariant derivative with respect to the connection ∇ and the canonical one-form that corresponds to the soldering

form on the frame bundle of M . On an n -dimensional orientable Riemannian manifold (M, g) , one may describe the torsion of a metric connection equivalently in terms of a one-form $A \in \Omega^1(M, so(n))$ via

$$\tau_{\nabla}(u, v) =: A(u)v - A(v)u \quad (u, v \in TM).$$

In particular, in the case of a Riemann surface the torsion of the most general metric connection reads

$$\tau_{\nabla} = A \otimes \mathfrak{I},$$

with $A \in \Omega^1(M_1)$ being an ordinary one-form. Accordingly, the torsion can be lifted to the (real) spinor bundle \mathbb{S} as $\frac{1}{2}A \otimes \gamma^1 \gamma^2 \in \Omega^1(M_1, \text{End}(\mathbb{S}))$.

Notice that the most general metric connection on M_1 reads

$$\begin{aligned} \nabla^g &= \nabla^{\text{LC}} + A \otimes \mathfrak{I} \\ &\stackrel{\text{loc.}}{=} d + (\Gamma + A) \otimes \mathfrak{I}, \end{aligned} \quad (25)$$

where $\Gamma(e_k) := g_M(\nabla_{e_k}^{\text{LC}} e_1, e_2)$ ($k = 1, 2$) is the connection form of the Levi-Civita connection with respect to an arbitrary (oriented) g_1 -orthonormal basis. We denote the induced connection on the (real) spinor bundle by $\partial_A := \nabla^S + \frac{1}{2}A \otimes \gamma^1 \gamma^2$, with $\nabla^S \stackrel{\text{loc.}}{=} d + \frac{1}{2}\Gamma \otimes \gamma^1 \gamma^2$ being the ordinary spin connection on \mathbb{S} .

In [11], it has been demonstrated how the functional

$$\int_{M_1} (\langle \psi, \not{A} \psi \rangle_{\mathbb{S} \otimes T^* M_1} + \|F_A\|^2) \, d\text{vol}(g_1)$$

can be derived from a specific class of Dirac operators. Here, respectively, \not{A} is the quantized connection ∂_A and $F_A := dA$ is the curvature form associated to the torsion τ_{∇} .

Even more, for smooth mappings $\varphi: M_1 \rightarrow M_2$ between arbitrary Riemann manifolds, it has been also shown in [11] how the functional of *Dirac harmonic maps with torsion*

$$\int_{M_1} (\text{scal}(g_1) + \|d\varphi\|^2 + \langle \psi, \not{A} \psi \rangle_{\mathbb{S} \otimes T^* M_1} + \|F_A\|^2) \, d\text{vol}(g_1). \quad (26)$$

naturally fits with the geometry of *Dirac operators of simple type*.

However, in the two-dimensional case, the part that involves the scalar curvature on M_1 becomes a topological invariant (Euler characteristic). Up to this constant, the functional (26) is known to be Weyl invariant if and only if $F_A = 0$. In this case, one may always find a gauge (i.e., a frame), such that torsion vanishes, locally (cf. (25)). Hence, by imposing conformal invariance (26) reduces to the functional (2) of Dirac harmonic maps. This demonstrates how the latter naturally fits with the geometry of Dirac operators of simple type. Let us point out, however, that by the geometrical construction indicated the (local) vanishing of torsion on Riemann surfaces is implemented by a co-homological condition instead put in by hand right-a-way. This co-homological condition may be viewed as the analogue of the independency of the functional of super harmonic maps (6) from the torsion, although torsion is around due to (11).

How does the gravitino enter the classical stage? The answer is provided by the factorization condition (12) imposed on the torsion by the demand of supersymmetry.

Indeed, one has the following

Proposition 1 *On a Riemann surface every $A \in \Omega^1(M_1)$ factorizes by a section $\chi \in \Gamma(M_1, \mathbb{S} \otimes T^* M_1)$, i.e.,*

$$A = \langle \gamma(\chi), \chi \rangle_{\mathbb{S}}. \quad (27)$$

Proof The proof basically takes advantage of the factorizations (17)–(18).

Explicitly, let A be given by $A = a_1 e^1 + a_2 e^2 = a\theta + \bar{a}\bar{\theta} \in \Omega^1(M_1)$ in terms of $\mathfrak{s} \in \mathfrak{Sec}(M_1, \mathbb{S})$. We may define the spinor \mathfrak{s} and the gravitino \mathfrak{g} in terms of A as follows

$$\begin{aligned}\mathfrak{s} &:= \operatorname{Re} \sqrt{a} \mathfrak{s}_1 - \operatorname{Im} \sqrt{a} \mathfrak{s}_2 = a_1 \mathfrak{s}_1 + a_2 \mathfrak{s}_2 \in \mathbb{S}, \\ \mathfrak{g} &:= \operatorname{Re} \frac{\bar{a}}{\sqrt{a}} (\mathfrak{s}_1 \otimes e^1 - \mathfrak{s}_2 \otimes e^2) + \operatorname{Im} \frac{\bar{a}}{\sqrt{a}} (\mathfrak{s}_2 \otimes e^1 + \mathfrak{s}_1 \otimes e^2) \\ &= \frac{1}{\|A\|} \left(\operatorname{Re} \bar{a}^{3/2} (\mathfrak{s}_1 \otimes e^1 - \mathfrak{s}_2 \otimes e^2) + \operatorname{Im} \bar{a}^{3/2} (\mathfrak{s}_2 \otimes e^1 + \mathfrak{s}_1 \otimes e^2) \right) \in \mathbb{G}.\end{aligned}$$

Apparently, the coefficients of the gravitino \mathfrak{g} transform with respect to a $3/2$ -representation of $\operatorname{Spin}(2) = \operatorname{SO}(2)$, whereas the coefficients of \mathfrak{s} with respect to the fundamental representation of the spin group. One calculates that

$$\langle \mathfrak{s}, \mathfrak{g} \rangle_{\mathbb{S}} = A.$$

Alternatively, one may write

$$\chi := \frac{1}{\sqrt{2}} (\delta_{\Theta} \mathfrak{s} + \mathfrak{g}) \in \Omega^1(M_1, \mathbb{S}),$$

such that

$$A = \langle \delta_{\gamma}(\chi), \chi \rangle_{\mathbb{S}}. \quad \square$$

Clearly, given $\chi \in \Omega^1(M_1, \mathbb{S})$, the formula (27) defines a torsion. Yet, the correspondence between the torsion and the gravitino (i.e., $A \leftrightarrow \chi$) is far from being unique. Of course, the assumption $F_A = 0$ (enforced by ordinary conformal symmetry) remedies this factorization ambiguity, for it guarantees that the functional (26) does not depend at all on torsion. Indeed there are many ways to factorize torsion in terms of the section χ . One may thus pose the question what is the invention of torsion good for when looking at the functional of Dirac harmonic maps from the classical point of view? In the language of physics, the answer can be expressed as follows. The factorization of torsion by gravitino fields allows to introduce additional “couplings” (i.e., invariants) in terms of the gravitino which would not appear otherwise. Of course, these additional invariants should be compatible with all the symmetries imposed. Especially, the couplings should obey conformal symmetry. When further restricted to at most quadratic invariants one ends up with the following three conformal invariants:

$$2\langle \psi, \operatorname{ev}(\chi \otimes \mathfrak{q}(\chi)) \psi \rangle_{\mathcal{E}} = \langle \chi_i, \gamma^j \gamma^i \chi_j \rangle_{\mathbb{S}} \langle \psi, \psi \rangle_{\mathcal{E}}, \quad (28)$$

$$2\langle \psi, (\chi \otimes \mathfrak{q}(\chi)) \psi \rangle_{\mathcal{E}} = \langle \chi_i, \gamma^j \gamma^i \psi^k \rangle_{\mathbb{S}} \langle \psi_k, \chi_j \rangle_{\mathbb{S}}, \quad (29)$$

$$2\langle \mathfrak{q}(\chi)(\operatorname{grad} \varphi), \psi \rangle_{\mathcal{E}} = \delta^{ij} \langle \chi_i \otimes \partial_j \varphi, \psi \rangle_{\mathcal{E}}. \quad (30)$$

Super Weyl invariance eventually allows to reduce these couplings further to exactly the two coupling terms that appear in the functional of super harmonic maps (6). Indeed, the conformal invariant (29) is ruled out by imposing invariance also under (7). We stress that the invariance of (28) under super Weyl transformations is guaranteed by the orthogonal decomposition (22).

Having clarified how the functional of super harmonic mappings may be motivated within the geometrical setup of classical differential geometry, we now turn to the question how to motivate the supersymmetry transformations (8)–(11) from this point of view. This is mainly due to the triviality of the Dirac action for real spinors, as will be discussed next.

4 The Dirac Action on Riemannian Surfaces and Supersymmetry

In the case of a (closed compact) two-dimensional Riemannian manifold, the Dirac action (3) vanishes in the case of Majorana (i.e., real) spinors. The reason for this is that the Clifford algebra $Cl_{2,0}$ has no skew-symmetric Majorana representations. In fact, the real spinor module only allows for symmetric representations of $Cl_{2,0}$. Hence, the standard spin Dirac operator \not{D} is skew-symmetric when acting on Majorana fields. Therefore, when switching to Dirac (i.e., complex) spinors, one may replace \not{D} by $i\not{D}$ to obtain a hermitian Dirac operator. However, in this case, the functional of super harmonic maps is not guaranteed to be real, which makes an analytical treatment of this functional more complicated.

The usual “way out” of this dilemma is to consider spinors as being “odd” (or “anti-commuting”) objects. More precisely, one assumes that there exist “superized” extensions $\hat{\mathbb{S}}$ of \mathbb{S} , \hat{g}_1 of the spinor metric g_1 , and $\hat{\omega}_1$ of ω_1 such that for all spinors $\hat{s}, \hat{s}' \in \hat{\mathbb{S}}$

$$\begin{aligned}\hat{g}_1(\hat{s}, \hat{s}') &= -\hat{g}_1(\hat{s}', \hat{s}), \\ \hat{\omega}_1(\hat{s}, \hat{s}') &= +\hat{\omega}_1(\hat{s}', \hat{s}).\end{aligned}\quad (31)$$

Note that the signs in (31) follow the rule that whenever two odd spinors are permuted an extra sign is acquired.

The main motivation for this sign rule is that it remedies the main cause for the vanishing of the Dirac action, namely the non-existence of skew-symmetric representations for $Cl_{2,0}$. It holds that every g_1 -symmetric Majorana representation of $Cl_{2,0}$ is $\hat{\omega}_1$ -skew-symmetric and vice versa:

$$g_1(s, \gamma(\alpha)s') = g_1(\gamma(\alpha)s, s') \Leftrightarrow \hat{\omega}_1(\hat{s}, \gamma(\alpha)\hat{s}') = -\hat{\omega}_1(\gamma(\alpha)\hat{s}, \hat{s}') \quad (32)$$

for all $\alpha \in TM_1$ and $s, s' \in \mathbb{S}$ (either “ordinary”, or “odd” spinors). Indeed, it is possible to obtain non-trivial Dirac actions using $\hat{\omega}_1$, see Lemma 1 below.

In super algebra, the vector bundle $\hat{\mathbb{S}}$ is constructed by extending the scalars to some anti-commuting ring. But, how can (31) be understood within the classical setting, thereby avoiding the notion of super algebra and super manifolds? A kind of “cheap” way in doing so is to assume that also the target manifold M_2 has a symplectic structure ω_2 and to use a twisted spinor bundle. More precisely, we replace φ^*g_2 by $\varphi^*\omega_2$ to obtain on the twisted spinor bundle

$$\pi_E: E := \mathbb{S}^* \otimes \varphi^*TM_2 \longrightarrow M_1 \quad (33)$$

the symmetric inner product

$$(\phi, \psi)_E = \epsilon^{kl} \omega_2(\phi_k, \psi_l) = (\psi, \phi)_E.$$

Here, the symplectic form ω_1^* on the dual spinor bundle is defined in terms of the dual of the Riemannian volume form on M_1 . Furthermore, we set $\epsilon^{kl} \equiv \omega_1^*(e^k, e^l)$ for any symplectic orthonormal basis $s^1, s^2 \in \mathbb{S}^*$.

As a matter of notation, we use the shorthand

$$\omega_2(\phi_k, \psi_l) \equiv \phi_k \cdot \psi_l,$$

such that formally the coefficients of the spinor fields $\psi = s^k \otimes \psi_k$ and $\phi = s^l \otimes \phi_l$ anti-commute, i.e.,

$$\phi_k \cdot \psi_l = -\psi_k \cdot \phi_l.$$

One then proves the following

Lemma 1 *For every Clifford connection on the twisted spinor bundle E (see (33)), the Dirac action (3) is a real-valued non-trivial functional. Furthermore, the Dirac action (3) does not depend on the metric connection used on the spinor bundle.*

Proof The prove of the statement relies crucially on (32). Consequently, for $k = 1, 2$, one gets

$$(\psi, \gamma^k \gamma^1 \gamma^2 \psi)_E = 0.$$

Hence,

$$(\psi, \not{d}_A \psi)_E = (\psi, \not{d} \psi)_E.$$

□

Consequently, the action functionals of Dirac harmonic maps (2) and super harmonic maps (6) can be realized also for the Clifford algebra $Cl_{2,0}$ using the twisted spinor bundle E instead of \mathcal{E} in the Dirac action. We put emphasize that already in this setting both functionals have all desired symmetry properties besides supersymmetry. In particular, the Dirac action does not vanish and the functional (6) is still real-valued as opposed to its supersymmetric analogue. Notice, however, that the realization of E provides a severe restriction on the target manifolds of the map φ . For example, it does not work in the most simple case $(M_2, g_2) = \mathbb{R}$.

We now want to check whether the action functional of Dirac harmonic map is invariant to the first order under the supersymmetry transformations

$$\begin{aligned} \delta_{\mathfrak{s}} \varphi &= \psi(\mathfrak{s}) = \omega_{\mathbb{S}^*}(\psi, \tilde{\mathfrak{s}}) \in \mathfrak{Sec}(M_1, \varphi^* T M_2), \\ \delta_{\mathfrak{s}} \psi &= \delta_{\gamma}(d\varphi) \tilde{\mathfrak{s}} \in \mathfrak{Sec}(M_1, E). \end{aligned} \quad (34)$$

Here, $\mathfrak{s} \in \Gamma(M_1, \mathbb{S})$ denotes an arbitrary spinor field and $\tilde{\mathfrak{s}} \in \Gamma(M_1, \mathbb{S}^*)$ its symplectic dual that is defined by $\tilde{\mathfrak{s}} := \omega_1(\mathfrak{s}, \cdot)$.

Notice that our convention for the symplectic dual is the following: for given symplectic orthonormal basis $\mathfrak{s}_1, \mathfrak{s}_2 \in \mathbb{S}$, with dual basis $\mathfrak{s}^1, \mathfrak{s}^2 \in \mathbb{S}^*$, the symplectic dual reads: $\tilde{\mathfrak{s}}_k := \omega_1(\mathfrak{s}_k, \cdot) = \epsilon_{kj} \mathfrak{s}^j \in \mathbb{S}^*$. Similarly, we define $\tilde{\mathfrak{s}}^l := \omega_1^*(\cdot, \mathfrak{s}^l) = -\epsilon^{ll} \mathfrak{s}_l \in \mathbb{S}$, such that for all $k, l = 1, 2$, we have $\tilde{\mathfrak{s}}^l(\tilde{\mathfrak{s}}_k) := \tilde{\mathfrak{s}}_k(\tilde{\mathfrak{s}}^l) = -\mathfrak{s}^l(\mathfrak{s}_k) = -\delta_k^l$.

To simplify the discussion, we restrict ourselves to the simple case of a trivial spinor bundle over $M_1 = \mathbb{C}$. Although the integral (2) is not well defined in this case, the variation of the integrand is of course.

Proposition 2 *Let \mathfrak{s}_0 be a constant spinor field. The variation of the functional of Dirac harmonic maps (2) that is defined to first order by the supersymmetry transformations (34) is given by*

$$\begin{aligned} \delta_{\mathfrak{s}_0} \|d\varphi\|^2 &= \delta^{kl} g_2(d\delta_{\mathfrak{s}_0} \varphi(e_k), d\varphi(e_l)) = -\langle \psi, \tilde{\mathfrak{s}}_0 \otimes \Delta \varphi \rangle_E + \operatorname{div} J_{\varphi}, \\ \delta_{\mathfrak{s}_0} (\psi, \not{d} \psi)_E &= \epsilon^{kl} \delta_{\mathfrak{s}_0} \psi_k \cdot \not{d} \psi_l = -(\psi, \tilde{\mathfrak{s}}_0 \otimes \Delta \varphi)_E + \operatorname{div} J_{\psi}, \end{aligned}$$

with the local tangent vector fields $J_{\varphi}, J_{\psi} \in \Gamma(M_1, T M_1)$ being given by

$$\begin{aligned} 2J_{\varphi} &:= \langle \psi, \tilde{\mathfrak{s}}_0 \otimes \operatorname{grad} \varphi \rangle_E, \\ 2J_{\psi} &:= -2(\psi, \Theta_x^{\sharp} \gamma(d\varphi) \mathfrak{s}_0)_E. \end{aligned}$$

Proof The statement is shown to hold true by a straightforward calculation. □

The statement clearly demonstrates that, using $(\cdot, \cdot)_E$ instead of $\langle \cdot, \cdot \rangle_E$ for the Dirac-term, the functional of Dirac-harmonic maps is not invariant with respect to the supersymmetry transformations (34) even for constant variational spinor fields. The detailed calculation demonstrates the failure in vanishing of the variation has its origin in the different inner products used on the twisted spinor bundle for the “bosonic” and “fermionic” parts of the functional of Dirac harmonic maps. Of course, this holds also true in the case of non-trivial spinor bundles and the more general functional (6) with supersymmetry transformations (8)–(11).

On the other hand, the only way to circumvent a trivial Dirac action on Riemann surfaces (for real spinors) consists in the usage of the symplectic form on the (real) spinor bundle instead of the usual metric. Again, this guarantees (symplectic) skew-symmetry of the Clifford action necessary for a symmetric Dirac operator.

We thus proceed with a geometrical construction very similar to what has been used in order to derive certain functionals, like (26), from Dirac operators of simple type (cf. [11]). The symmetry property (31) alone has not proven sufficient to reproduce super symmetry. Therefore, we specify more properties of odd spinors and give another construction of $(\cdot, \cdot)_E$. That is, for given Grassmann algebra Λ , we consider the *Grassmann extension* of the twisted spinor bundle (33)

$$E_\Lambda := \mathbb{S}^* \otimes \varphi^* T M_2 \otimes \Lambda \longrightarrow M_1. \quad (35)$$

Similarly, one may replace the ordinary tangent bundle by its Grassmann extension $T M_1 \otimes \Lambda \rightarrow M_1$, etc..

Clearly, the Grassmann extension of any vector bundle contains the latter as a distinguished sub-vector bundle. For instance, (35) contains (33) because of the canonical embedding

$$\begin{aligned} E &\hookrightarrow E_\Lambda \\ \mathfrak{z} &\longmapsto \mathfrak{z} \otimes 1. \end{aligned}$$

Of course, this holds true similarly for every Grassmann extension. Accordingly, by a slight abuse of notation, we do not make a distinction between a vector bundle and its Grassmann extension. That is, for given Λ , every (real) vector bundle is considered to be contained into its Grassmann extension. By construction, all vector bundles are naturally \mathbb{Z}_2 -graded. By an “even/odd” section, we thus mean a section restricted to Λ^\pm .

We consider the following graded inner (fiber) product with values in the \mathbb{Z}_2 -graded ring Λ :

$$\begin{aligned} (\cdot, \cdot)_E: \quad E \times_M E &\longrightarrow \Lambda \\ (\mathfrak{z}_1 \otimes \lambda_1, \mathfrak{z}_2 \otimes \lambda_2) &\longmapsto \langle \mathfrak{z}_1, \mathfrak{z}_2 \rangle_E \lambda_1 \wedge \lambda_2. \end{aligned} \quad (36)$$

Here, for all homogeneous elements $\mathfrak{z}_k = \mathfrak{s}_k^* \otimes y_k \in E$ ($k = 1, 2$), we put

$$\langle \mathfrak{s}_1^* \otimes y_1, \mathfrak{s}_2^* \otimes y_2 \rangle_E := \omega_1^*(\mathfrak{s}_1^*, \mathfrak{s}_2^*) \varphi^* g_2(y_1, y_2) \in \mathbb{R}.$$

It follows that for all elements of definite parity

$$(\mathfrak{z}, \mathfrak{w})_E = -(-1)^{|\mathfrak{z}||\mathfrak{w}|} (\mathfrak{w}, \mathfrak{z})_E.$$

Lemma 2 *The twisted spin-Dirac operator $\not{D} = \delta_Y(\nabla^{\mathbb{S} \otimes \varphi^* T M_2})$, restricted to odd sections, is symmetric with respect to the scalar product*

$$\begin{aligned} ((\phi, \psi)) &:= \int_{M_1} (\phi, \psi)_E d\text{vol}(g_1) \\ &= ((\psi, \phi)) \in \Lambda^+. \end{aligned} \quad (37)$$

Proof The proof is straightforward and makes use of $\not\partial$ being a quantized Clifford connection. Hence,

$$\nabla^{T^*M_1 \otimes \mathbb{S} \otimes \varphi^* TM_2} \Theta = 0.$$

Furthermore, the Clifford action is skew-symplectic. Therefore,

$$(\Theta^\sharp \phi, \psi)_E + (\phi, \Theta^\sharp \psi)_E = 0.$$

Altogether the calculation yields

$$\operatorname{div} J = -(\not\partial \phi, \psi)_E + (\phi, \not\partial \psi)_E,$$

with the (even) tangent vector field $J \in \Gamma(M_1, TM_1)$ being given by

$$J := 2(\phi, \Theta^\sharp \psi).$$

□

As a consequence, the Dirac action for odd (twisted) Majorana spinors

$$((\psi, \not\partial \psi)) = \int_{M_1} (\psi, \not\partial \psi)_E \, d\operatorname{vol}(g_1) \in \Lambda^+$$

is non-trivial as opposed to the case of even Majorana spinors. Similar to Lemma 1, one may prove that the Dirac action is actually independent of the metric connection used on the spinor bundle.

Proposition 3 *Assume that the odd variational spinor field $\mathfrak{s} \in \Gamma_{\operatorname{hol}}(M_1, \mathbb{S}^-)$ is holomorphic and (M_2, g_2) be flat. The functional*

$$\mathcal{A}_{\operatorname{SDH}}(\varphi, \psi) := \int_{M_1} (\|d\varphi\|^2 + (\psi, \not\partial_A \psi)_E) \, d\operatorname{vol}(g_1) \in \Lambda^+ \quad (38)$$

is stationary with respect to the variation determined to first order by the supersymmetry transformations (34).

Proof As the integrand in (38) is conformally invariant, we are allowed to work in coordinates x^1, x^2 on M_1 such that $g_1 = (dx^1)^2 + (dx^2)^2$. Furthermore, as the integrand does not depend on A , we will work with $A = 0$. It follows that in the specific coordinates chosen, the covariant derivative can be expressed as $d_\nabla \mathfrak{s} = d\mathfrak{s}$. For arbitrary but fixed $t \in M_1$, we denote the (oriented) orthonormal frame $\partial_{x^1}, \partial_{x^2}$ by $e_1, e_2 \in T_t M_1$ and consider the following Λ -valued smooth mappings:

$$\begin{aligned} \|d\varphi_t\|^2 &: (-\varepsilon, \varepsilon) \longrightarrow \Lambda^+ \\ s &\longmapsto \delta^{ij} \varphi^* g_2(d\Phi(s, t)e_i, d\Phi(s, t)e_j), \\ (\psi_t, \not\partial_A \psi_t)_E &: (-\varepsilon, \varepsilon) \longrightarrow \Lambda^+ \\ s &\longmapsto \omega_1^*(\mathfrak{s}^i, \gamma^k \mathfrak{s}^j) \varphi^* g_2(\Psi_i(s, t), d\Psi_j(s, t)e_k), \end{aligned}$$

whereby to first order for all $v \in T_t M_1$

$$\begin{aligned} d\Phi(s, t)v &:= d\varphi(t)v + sd\psi(\mathfrak{s})(t)v + o(s) \in T_{\varphi(t)} M_2, \\ \Psi(s, t) &:= \psi(t) + s\gamma(d\varphi(t))\tilde{\mathfrak{s}}(t) + o(s) \in E_t^-. \end{aligned}$$

Accordingly, one obtains

$$\begin{aligned}\frac{d\|d\varphi_t\|^2}{ds}(s)|_{s=0} &= -2(\psi, \Delta\varphi(t) \otimes \tilde{s}(t))_E + \operatorname{div} \mathfrak{J}_\varphi(t), \\ \frac{d(\psi_t, \not\partial_A \psi_t)}{ds}(s)|_{s=0} &= 2(\psi, \Delta\varphi(t) \otimes \tilde{s}(t))_E - 2(\psi(t), \gamma^k \gamma(d\varphi(t)) d\mathfrak{s}(t) e_k(x))_E \\ &\quad + 2\operatorname{div} \mathfrak{J}_\psi(t),\end{aligned}$$

with the even vector fields $\mathfrak{J}_\varphi, \mathfrak{J}_\psi \in \Gamma(M_1, TM_1)$ being given by

$$\begin{aligned}\mathfrak{J}_\varphi &:= 2(\psi, \operatorname{grad} \varphi \otimes \mathfrak{s})_E, \\ \mathfrak{J}_\psi &:= 4(\psi, \Theta^\sharp \gamma(d\varphi) \mathfrak{s})_E.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d\|d\varphi_t\|^2}{ds}(s)|_{s=0} + \frac{d(\psi_t, \not\partial_A \psi_t)}{ds}(s)|_{s=0} &= -2(\psi(t), \gamma^k \gamma(d\varphi(t)) d\mathfrak{s}(t) e_k(t))_E \\ &\quad + 2\operatorname{div} \mathfrak{J}_{\text{susy}},\end{aligned}$$

with the “susy-current” $\mathfrak{J}_{\text{susy}} \in \Gamma(M_1, TM_1)$ reading

$$\mathfrak{J}_{\text{susy}} := (\psi, \operatorname{grad} \varphi \otimes \mathfrak{s})_E + 4(\psi, \Theta^\sharp \gamma(d\varphi) \mathfrak{s})_E.$$

Since

$$2\gamma^k \gamma(d\varphi) d\mathfrak{s}(e_k) = \mathfrak{q}(d\mathfrak{s}) \operatorname{grad} \varphi,$$

the statement follows from (23). Indeed, with $\theta = dz$, for the holomorphic coordinate $z = x^1 + ix^2$ and $\mathfrak{s} = u\mathfrak{w}$, it holds that

$$0 = \mathfrak{q}(d\mathfrak{s}) = \mathfrak{q}((\partial_z u)\theta \otimes \mathfrak{w} + (\partial_{\bar{z}} u)\bar{\theta} \otimes \mathfrak{w}) = (\partial_{\bar{z}} u)\bar{\theta} \otimes \mathfrak{w}.$$

Notice that even though the last equation is obtained using particular local coordinates, due to conformal invariance, it glues to the global condition that \mathfrak{s} be holomorphic. \square

The statement of Proposition 3 can be generalized, actually. By a similar but (much) more involved calculation (analogous to the corresponding calculation presented in all details in [9], see also [1]), one may finally prove the following

Proposition 4 *Let, again, (M_2, g_2) be flat. Also, let $\mathfrak{s} \in \Gamma(M_1, \mathbb{S}^-)$ be an arbitrary odd spinor field and $\tilde{\mathfrak{s}} \in \Gamma(M_1, \mathbb{S}^*)$ its symplectic dual. The functional*

$$\begin{aligned}\mathcal{A}_{SRS}(\varphi, \psi, \chi, g_1) &:= \int_{M_1} (\|d\varphi\|^2 + (\psi, \not\partial_A \psi)_E + (\chi, \mathfrak{q}(\chi))_{T^*M \otimes \mathbb{S}}(\psi, \psi)_E \\ &\quad + 4(\mathfrak{q}(\chi)(\operatorname{grad} \varphi), \psi)_E) \operatorname{vol}(g_1),\end{aligned}\tag{39}$$

with $\chi \in \Omega^1(M_1, \mathbb{S}^-)$, is stationary to first order with respect to

$$\begin{aligned}\delta_{\mathfrak{s}} \varphi &:= \psi(\mathfrak{s}) \in \Gamma(M_1, \varphi^* TM_2), \\ \delta_{\mathfrak{s}} \psi &:= \delta_\gamma(d\varphi - \operatorname{ev}(\psi, \chi)) \tilde{\mathfrak{s}} \in \Gamma(M_1, E^-), \\ \delta_{\mathfrak{s}} e_k &:= -4\omega_1^*(\delta_{\Theta^\sharp \mathfrak{s}}, \chi(e_k)) \in \Gamma(M_1, TM_1) \quad (k = 1, 2), \\ \delta_{\mathfrak{s}} \chi &:= d_A \mathfrak{s} \in \Omega^1(M_1, \mathbb{S}^-),\end{aligned}$$

if and only if the torsion factorizes as

$$A = \omega_1(\delta_\gamma(\chi), \chi).$$

The realization of anti-commutative spinors (31) by Grassmann extension of ordinary spinors thus allows a “semi-classical” interpretation of the functional of super harmonic maps and its symmetries, including supersymmetry. The prize to be paid is to deal with functionals taking values in a non-commutative ring instead of being real-valued. This is rather close, indeed, to super geometry. Indeed, one can show that the action functional $\mathcal{A}_{SR\mathcal{S}}$ from (39) is equivalent to the action functional \mathcal{I}_{SDH} studied in [8] under the assumption of a trivial family of super Riemann surfaces. It seems unavoidable to use the full language of super geometry, in particular ringed spaces involving anti commutative variables, and reminiscent to the supersymmetry transformations, which seem to have a clear geometrical interpretation only within the realm of super geometry.

In contrast, the functional (6) and the factorization (12) of the torsion were shown to be already well-motivated by pure symmetry considerations also within the realm of ordinary differential geometry.

Acknowledgments The research leading to these results has received funding from the European Research Council under the European Union’s Seventh Framework Program (FP7/2007–2013)/ERC grant agreement n°267087.

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

1. Brink, L., di Vecchia, P., Howe, P.: A locally supersymmetric and reparametrization invariant action for the spinning string. *Phys. Lett. B* **65**, 471–474 (1976)
2. Chen, Q., Jost, J., Li, J., Wang, G.: Dirac-harmonic maps. *Math. Z.* **254**, 409–432 (2006)
3. Chen, Q., Jost, J., Li, J., Wang, G.: Regularity theorems and energy identities for Dirac-harmonic maps. *Math. Z.* **251**, 61–84 (2006)
4. Deser, S., Zumino, B.: A complete action for the spinning string. *Phys. Lett. B* **65**, 369–373 (1976)
5. D’Hoker, E., Phong, D.H.: The geometry of string perturbation theory. *Rev. Mod. Phys.* **60**, 917–1065 (1988)
6. Eells, Jr., J., Sampson, J.H.: Harmonic mappings of Riemannian manifolds. *Am. J. Math.* **86**, 109–160 (1964)
7. Jost, J.: *Riemannian Geometry and Geometric Analysis*. Springer, Berlin (2011)
8. Jost, J., Keßler, E., Tolksdorf, J.: *Super Riemann surfaces, metrics, and gravitinos* (2014). [arXiv:1412.5146](https://arxiv.org/abs/1412.5146)
9. Keßler, E.: PhD Thesis. Universität Leipzig, Leipzig (2016). In preparation
10. Sachse, C.: *Global Analytic Approach to Super Teichmüller Spaces*. PhD Thesis. Universität Leipzig, Leipzig (2007)
11. Tolksdorf, J.: The Einstein-Hilbert action with cosmological constant as a functional of generic form. *J. Math. Phys.* **56** (2015)